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Theory and Methodology

An amalgation of games

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Abstract

A special class of normal form games is the subject of the paper. Typically, the player set of the games in this class consists of two parties and the games are aggregations of conflicts between two players, one in each party. Two 2-person normal form games, closely related to the original game, are introduced and relations between the sets of (perfect) equilibria of these games and the original game are derived. Using the fact that the structure of the set of (perfect) equilibria of bimatrix games is known, the structure of the set of (perfect) equilibria of the original games is revealed. Two characteristic examples are treated in more detail.

Keywords: Perfect equilibrium; Nash component; Selten component; Correlation; Coordination

1. Introduction

Daily life conflicts are mostly less well-described and more ambiguous than game theoretists would like. In the 'game of life' a player often has to use the same strategy (behavior) in the conflicts with all his possible opponents. Closing your door during the night protects your family against a possible attempt of *any* member of the 'guild of thieves' to steal your properties. In nature prey animals have to choose a way of behavior that protects them against *all* predators living in their habitat. A tax inspector has to find a strategy applied to *all* tax payers in his region.

These examples have in common that the 'real conflict' is an amalgation of 'private conflicts between individuals' and that the players can only use one and the same strategy in all their conflicts.

In this paper we therefore take the following position:

(a) The player set has a partitioning into two (or more) parties.
(b) Each player of one party has a 'private conflict' with each member of the other party. This private conflict is modeled by a bimatrix game. [1]

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[1] If there are $p > 2$ parties the private conflicts are mixed extensions of finite p-person games between representants of each of the parties.

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If these four conditions are satisfied, we call the conflict an *amalgation of games*.

With an amalgation of games we associate two 2-person games, the *coordination game* and the *correlation game*. In both games there are two players (or as many as there are parties) and the players are aiming at a maximal total payoff for the group they belong to. In coordination games each player is subcoordinating his strategy choice to the social interests of his party. The conflict is now between parties and not between individuals.

In the correlation game the cooperation is more intense. The group as a whole is determining a probability distribution over the pure strategy *combinations* of their members. The difference between coordination games and correlation games is similar to the difference between behavior strategies and general mixed strategies in extensive form games. Note that coordinating strategies requires a (common knowledge) signalling system to perform the coordination e.g. blowing the horn for a coordinated attack to the enemy or a combination of higher water temperature and high tide as a signal for a coordinated mating behaviour of many turtle species.

In this paper we investigate the relations between the equilibrium set of an amalgation of games (a many-person game) and the equilibrium sets of its coordination and correlation game (2-person games). The same we do for the set of perfect equilibria of those three games. To obtain this result we need an alternative characterization of prefect equilibria and the equivalence of perfectness and undominatedness for 2-person games with *polytopal* strategy spaces and bilinear payoff functions. From the close relation between these equilibrium sets we come to a description of the set of (perfect) equilibria of amalgations of games. We finish the paper with a more explicit elaboration of two examples. These examples will exhibit three – we think, general – features of amalgations:

(i) Nash equilibrium behavior leads automatically to a coordination of strategies. This phenomenon can be seen as an application of the adagium "The enemies of my enemies are my friends". Having the same opponents gives rise to a 'natural kind of solidarity'.

(ii) Correlating strategies doesn't help in the sense that it does not change the possible equilibrium payoffs.

(iii) Compared with the private conflicts there is a 'de-mixing' of strategies. Instead of playing a mixed strategy the roles are distributed among the members of a party. One of the members of the party plays one pure strategy and an other member takes a second pure strategy. This is what happens when forming a community: Not everybody is performing every task some part of his time but he is doing some tasks all of the time.

Before we start a formal description of this kind of games we will discuss two typical examples.

The tax inspector

A tax inspector has a private conflict with each of the tax payers. The pure strategies of the tax inspector are checking (C) or non-checking (N) the declaration of a client. The tax payer has also two pure strategies: being honest (H) or dishonest (D). If $c > 0$ is the amount of tax that the tax payer may try to evade, $\gamma > 0$ is the cost of checking and $f > 0$ (maybe dependent on c) is the fine the tax payer has to pay if he is trapped, the following bimatrix game models the private conflict:

	C	N
D	$-f, f-\gamma$	$c, -c$
H	$0, -\gamma$	$0, 0$

If $f + c > \gamma$, the (unique) Nash equilibrium is $((p_D, p_H), (q_C, q_N))$ with

$$p_D = \frac{\gamma}{f+c} \quad \text{and} \quad q_C = \frac{c}{f+c},$$

leading to payoffs 0 and $-c\gamma/(f+c)$ respectively.

By the choice of his strategy the tax inspector makes the tax payer indifferent between being Honest and being Dishonest. In reality the tax inspector can only use the same checking rate q_C for all tax payers (as he doesn't know the number c and has only some statistical evidence about how many people try to evade how much). Therefore he is forced to play one and the same strategy against the whole population of tax payers. In Section 5 we will have a closer look at this example.

Prey contra predator

In an ecological system there are predators and prey animals, say wolves and eagles vs. rabbits and mice. The predators have to choose a time and place for hunting and the preys must decide at what time and place they go for foraging. Different choices of their strategies give the wolves and eagles different quantities of necessary proteins (and thereby chance of surviving). The choice of a good time (and place) for foraging gives the preys an undisturbed meal and therewith enough nutricients. The preys try to evade all the predators by the choice of their strategy and the predators are not hunting for rabbits or mice but just hunting. Here again we have an amalgation of conflicts (games).

In the literature one finds many applications of game theory to sociobiology. The best known is Maynard Smith's book 'Evolution and the theory of games' (1982) (see also Maynard Smith and Price, 1973). One of the main topics of this book explains how stable animal behavior can emerge from an ocean of possible deviating behaviors. The game theoretical models in the book are symmetric (every individual has the same possibilities) and describe an *intra-species conflict* between animals with equilibrium behavior and animals (of the same species) with deviating behavior. The example we give describes an *inter-species conflict* and does not show any symmetry. A more recent paper with a sociobilogical application is McNamara and Collins (1990). [2] In this paper we find a repeated game that explains the way animals (e.g. birds) choose a mating partner by a series of inspections. This is rather far from the model we give.

Polymatrix games in the sense of Quintas (1989) and Yanovskaya (1968) are closely related to the type of games we are considering. In these games we also have an aggregation of 2-person conflicts but every player has a conflict with every other player. The player set is not bipartite as in our model.

2. The main features of the model

Let $\mathscr{A}$ be a strategic game of the following form. The player set has a bipartite structure $M \cup N$. Each player $i \in M$ or $j \in N$ has a finite set of pure strategies $E^i := \{e_1^i, \ldots, e_{m^i}^i\}$ or $F^j := \{f_1^j, \ldots, f_{n^j}^j\}$ respectively. Each player i in M has a 'private conflict' with each player j in N. This conflict is modelled by a bimatrix game $(X^i, Y^j, K^{i,j}, L^{i,j})$ where X^i and Y^j are the probability vectors on the sets E^i and F^j respectively. In the 'global game' the strategy spaces are the same and the payoff function for player $i \in M$ is the sum of the payoff functions in the 'private conflicts' with players $j \in N$ i.e., $K^i(x, y) = \Sigma_{j \in N} K^{i,j}(x^i, y^j)$ and, similarly, $L^j(x, y) = \Sigma_{i \in M} L^{i,j}(x^i, y^j)$ for players $j \in N$. Notice that the payoff to each player only depends on his strategy and the strategies of his opponents (so sometimes we write $K^i(x^i, y)$ or $L^j(x, y^j)$ instead of $K^i(x, y)$ or $L^j(x, y)$), and each player has to use the *same strategy* in all his private conflicts. We call a strategic game with these properties an *amalgation of games*.

For an amalgation of games $\mathscr{A}$ we introduce the associated *coordination game* $\Gamma(\mathscr{A})$. It has two players $[M]$ and $[N]$ (the parties as a whole) and strategy spaces $\prod_{i \in M} \Delta(E^i)$ and $\prod_{j \in N} \Delta(F^j)$. [3]

[2] We thank the referee who pointed us to this paper.

[3] With $\Delta(X)$ we mean the probability space on the finite set X.

The payoff functions K and L of player $[M]$ and $[N]$ respectively are defined by
$K(x, y) := \sum_{i \in M} K^i(x^i, y) = \sum_{i \in M} \sum_{j \in N} K^{i,j}(x^i, y^j)$ and
$L(x, y) := \sum_{j \in N} L^j(x, y^j) = \sum_{j \in N} \sum_{i \in M} L^{i,j}(x^i, y^j)$, respectively.
The associated *correlation game* $G(\mathscr{A})$ has the same two players but the strategy sets are now $\Delta(\prod_{i \in M} E^i)$ and $\Delta(\prod_{j \in N} F^j)$. We denote the extreme points of the first set by $e_\alpha = (e^i_{\alpha(i)})_{i \in M}$ with $1 \le \alpha(i) \le m^i$. The extreme points of the strategy space of player $[N]$ are denoted by f_β. The payoff functions $\bar{K}$ and $\bar{L}$ are linear in each of the two strategies and for pure strategies $e_\alpha \in \prod_{i \in M} E^i$ and $f_\beta \in \prod_{j \in N} F^j$ we define $\bar{K}(e_\alpha, f_\beta) := \sum_{i \in M} K^i(e^i_{\alpha(i)}, f_\beta)$, and $\bar{L}(e_\alpha, f_\beta)$ is defined in a similar way.

To compare strategies of $\Gamma(\mathscr{A})$ and $G(\mathscr{A})$ we introduce the *marginal maps*

$$\mu : \Delta\Big(\prod_{i \in M} E^i\Big) \to \prod_{i \in M} \Delta(E^i) \quad \text{and} \quad \nu : \Delta\Big(\prod_{j \in N} F^j\Big) \to \prod_{j \in N} \Delta(F^j)$$

by $\mu(u) = x$ if for all $i \in M$ and for all $p \in \{1, \ldots, m^i\}$

$$x^i_p = \sum_{\alpha : \alpha(i) = p} u_\alpha \quad \text{if } u = \sum_\alpha u_\alpha e_\alpha \in \Delta\Big(\prod_{i \in M} E^i\Big).$$

The marginal map ν is defined similarly. Notice that μ and ν are linear and subjective maps. Partial inverses are the maps $\bar{\mu}$ and $\bar{\nu}$ with $\bar{\mu}(x) = u$ if $u_\alpha = \prod_{i \in M} x^i_{\alpha(i)}$ for all α. The map $\bar{\nu}$ is defined analogously.

Between the payoff functions in the coordination game and the payoff functions in the correlation game we have the following relations:

$$\bar{K}(u, v) = K(\mu(u), \nu(v)) \quad \text{and} \quad \bar{L}(u, v) = L(\mu(u), \nu(v)).$$

This is clear, as

$$\begin{aligned}
\bar{K}(u, v) &= \sum_\alpha \sum_\beta u_\alpha v_\beta \bar{K}(e_\alpha, f_\beta) && \text{(linearity)} \\
&= \sum_\alpha \sum_\beta u_\alpha v_\beta \sum_{i \in M} K^i\big(e^i_{\alpha(i)}, f_\beta\big) && \text{(definition of } \bar{K}) \\
&= \sum_\alpha \sum_\beta u_\alpha v_\beta \sum_{i \in M} \sum_{j \in N} K^{ij}\big(e^i_{\alpha(i)}, f^j_{\beta(j)}\big). && \text{(additivity of } K^i)
\end{aligned}$$

If we collect the terms with $K^{ij}(e^i_p, f^j_q)$, we find

$$\bar{K}(u, v) = \sum_{i \in M} \sum_{j \in N} \Bigg(\sum_{\alpha : \alpha(i) = p} u_\alpha \Bigg) \Bigg(\sum_{\beta : \beta(j) = q} v_\beta \Bigg) K^{ij}\big(e^i_p, f^j_q\big).$$

Hence we find, with $x = \mu(u)$ and $y = \nu(v)$,

$$\begin{aligned}
\bar{K}(u, v) &= \sum_{i \in M} \sum_{j \in N} x^i_p y^j_q K^{ij}\big(e^i_p, f^j_q\big) && \text{(definition of } \mu \text{ and } \nu) \\
&= K(\mu(u), \nu(v)).
\end{aligned}$$

In Table 1 we summarize the various notations.

Notice that the total strategy space of the games $\mathscr{A}$ and $\Gamma(\mathscr{A})$ are the same:

$$\underbrace{X^1 \times \cdots \times X^m}_{[M]} \times \underbrace{Y^1 \times \cdots \times Y^n}_{[N]}.$$

Therefore, the statement $E(\mathscr{A}) = E(\Gamma(\mathscr{A}))$ in the next proposition makes sense.

Table 1

Player sets	M, $\lvert M \rvert = m$	N, $\lvert N \rvert = n$
Sets of (mixed) strategies	$i \in M: X^i = \Delta(E^i)$	$j \in N: Y^j = \Delta(F^j)$
	$\lvert E^i \rvert = m^i$, $E^i = \{e_1^i, \ldots, e_{m^i}^i\}$	$\lvert F^j \rvert = n^j$, $F^j = \{f_1^j, \ldots, f_{n^j}^j\}$
Payoff functions in private conflicts	K^{ij}	L^{ij}
Payoff function in $\mathscr{A}$	$K^i := \sum_{j \in N} K^{ij}$	$L^j := \sum_{i \in M} L^{ij}$
Payoff function in $\Gamma(\mathscr{A})$	$K := \sum_{i \in M} K^i$	$L := \sum_{j \in N} L^j$
Payoff function in $G(\mathscr{A})$	$\overline{K}$	$\overline{L}$
The marginal map and its inverse	μ, $\overline{\mu}$	ν, $\overline{\nu}$
The pure strategies in $G(\mathscr{A})$	$e_\alpha = (e_{\alpha(1)}^1, \ldots, e_{\alpha(m)}^m)$	$f_\beta = (f_{\beta(1)}^1, \ldots, f_{\beta(n)}^n)$
Mixed strategies in $\mathscr{A}$, $\Gamma(\mathscr{A})$ and $G(\mathscr{A})$	$x = (x^1, \ldots, x^m)$, $u = \{u_\alpha\}$	$y = (y^1, \ldots, y^n)$, $v = \{v_\beta\}$

In the next proposition we give a relation between the equilibrium sets $E(\mathscr{A})$, $E(\Gamma(\mathscr{A}))$ and $E(G(\mathscr{A}))$. It says that the set of equilibria of the games $\mathscr{A}$ and $\Gamma(\mathscr{A})$ are the same and that the marginal map (μ, ν) maps the equilibrium set of $G(\mathscr{A})$ to the equilibrium set of $\Gamma(\mathscr{A})$.

Proposition 1. (i) $E(\mathscr{A}) = E(\Gamma(\mathscr{A}))$.
(ii) $(u, v) \in E(G(\mathscr{A}))$ *if and only if* $(\mu(u), \nu(v)) \in E(\Gamma(\mathscr{A}))$.

Proof. (i): If (x, y) is a strategy pair in the game $\Gamma(\mathscr{A})$ and player $[M]$ has an advantageous deviation from x to $\bar{x}$, then $\sum_{i \in M} K^i(x^i \mid y) < \sum_{i \in M} K^i(\bar{x}^i \mid y)$. This gives $K^i(x^i \mid y) < K^i(\bar{x}^i \mid y)$ for at least one player $i \in M$. Therefore player i has also an advantageous deviation in $\mathscr{A}$. The converse is evident since an advantageous deviation from x^i to $\bar{x}^i$ of player $i \in M$ gives an advantageous deviation from x to $\bar{x} = (\bar{x}_i \mid x_{-i})$ for player $[M]$ in $\Gamma(\mathscr{A})$.

(ii): If (u, v) is a strategy pair in $G(\mathscr{A})$ and it is profitable to deviate from u to $\bar{u}$, then

$$K(x, y) = \overline{K}(u, v) < \overline{K}(\bar{u}, v) = K(\bar{x}, y) \quad \text{if } x = \mu(u),\ y = \nu(v) \text{ and } \bar{x} = \mu(\bar{u}).$$

Then the deviation from x to $\bar{x}$ is also profitable in $\Gamma(\mathscr{A})$.

Conversely, if (x, y) is a strategy pair of $\Gamma(\mathscr{A})$ and the deviation from x to $\bar{x}$ is profitable, if u, $\bar{u}$ and v have marginals x, $\bar{x}$ and y (e.g. if $u = \overline{\mu}(x)$, $\bar{u} = \overline{\mu}(\bar{x})$ and $v = \overline{\nu}(y)$), then

$$\overline{K}(u, v) = K(x, y) < K(\bar{x}, y) = \overline{K}(\bar{u}, v),$$

which implies that (u, v) is not an equilibrium in $G(\mathscr{A})$. □

Remark 1. If we represent the game $\mathscr{A}$ by an extensive form game (each player has one information set and doesn't know the decisions of the players earlier in the decision tree), we obtain an extensive form representation of $G(\mathscr{A})$ if we take the players in the same party (M or N) as agents of one player and add the payoffs of the players in each party. If we restrict the strategies in this extensive form game to *behavior strategies* we obtain an extensive form representation of $\Gamma(\mathscr{A})$. The extensive form game of $G(\mathscr{A})$ is *not* a game with *perfect recall*. Therefore the Theorem of Kuhn (1958) cannot be applied. Nevertheless we showed that restriction to behavior strategies doesn't lead to a restriction on the equilibrium payoffs.

Remark 2. Amalgations of games are closely related to 2-person games with *incomplete information on both sides*. There is hardly any difference between the situation that a player $i \in M$ has to fight against all players in N and the situation that the player has to fight against one randomly chosen opponent from N. Hence, the results of this paper will also have impact on two-sided incomplete information models.

Remark 3. Although Proposition 1 gives a rather simple result, it reveals two important features of the games we are considering. Simply by playing a Nash equilibrium the players of the same party are coordinating their strategies. Having the same enemies gives rise to a natural kind of solidarity. Furthermore, it is not worthwhile to try (complicated) correlated strategies.

3. Perfect equilibria

In this section we will prove that the results we obtained for equilibria in Section 2, also hold for perfect equilibria. First we must extend the perfectness concept to a larger class of games as coordination games are not bimatrix games but more general two-person games with bilinear payoff functions. There are several (equivalent) definitions of perfectness in the theory of mixed extensions of finite n-person games. Classically (see Selten, 1975) an equilibrium is perfect if it is the limit of ε-perfect strategy n-tuples for ε converging to zero. We take however the following one that can easily be extended to other games with (multi)linear payoff functions.

Let G be a strategic game $\langle N, \{P^i\}_{i\in N}, U=\{U^i\}_{i\in N}\rangle$ where P^i are *polytopes* and $U: \prod_{i\in N} P^i \to \mathbb{R}^N$ is a map, linear in each of the coordinates. For games G we will define the perfectness concept. First we define the kind of *perturbations* we consider. If P is a polytope, x_0 is a point in the relative interior of P and η is a number in the interval (0, 1), we define

$$P(x_0, \eta) := \{(1-\eta)x + \eta x_0 \mid x \in P\}.$$

A *perturbation* of a game $G = \langle N; \{P^i\}_{i\in N}, U\rangle$ is determined by a point $\bar{x}_0 = (x_0^1, \dots, x_0^n)$ with x_0^i in the relative interior of P^i for all $i \in N$ and $\eta \in (0, 1)$. The *perturbed game* $G(\bar{x}_0, \eta)$ is the game $\langle N, \{P^i(x_0^i, \eta)\}_{i\in N}, U\rangle$. The perturbed game has the same player set and payoff function as G but only a subset of the strategies of G.

Definition. A strategy profile $\hat{x} = (\hat{x}^1, \dots, \hat{x}^n)$ is a *perfect equilibrium of the game* G if for every open neighborhood V of $\hat{x}$ and every number $\delta > 0$, *there is* a perturbation $(\bar{x}_0, \eta)$ with $\eta > \delta$ and $E(G(\bar{x}_0, \eta)) \cap V \neq \emptyset$.

Note that the class of perturbations we admit is *smaller* than usually. In particular, we require that η is the same for all players. In the Appendix of Borm et al. (1993) we proved that the definition of perfectness we gave and the classical definition agree for n-matrix games.

Now we can formulate the main result of this section.

Proposition 2. *If $\mathscr{A}$ is an amalgation of games, then* (i) *the set of perfect equilibria of $\mathscr{A}$ and $\Gamma(\mathscr{A})$ are the same, and* (ii) *(u, v) is a perfect equilibrium of $G(\mathscr{A})$ if and only if $(\mu(u), \nu(v))$ is a perfect equilibrium of Γ $(\mathscr{A})$.*

In the proof of the proposition we use the equivalence of undominatedness and perfectness for equilibria of 2-person games of the kind we are talking about. First we repeate the definition of undominatedness.

A strategy $x^i \in P^i$ is *undominated* if there is no strategy $y^i \in P^i$ such that $U^i(y^i \mid x^{-i}) \geq U^i(x^i \mid x^{-i})$ *for all* $x^{-i} \in \prod_{j\neq i} P^j$ and there is a strict inequality for at least one x^{-i}. An equilibrium is called *undominated* if all players use an undominated strategy.

In Van Damme (1991) it is proved that perfect equilibria are undominated and that for *bimatrix games* the converse also holds.

This means that PF($G(\mathscr{A})$) = UN($G(\mathscr{A})$), if PF and UN denote the set of perfect and undominated equilibria, respectively. In Borm et al. (1993) it is proved that PF($\Gamma(\mathscr{A})$) = UN($\Gamma(\mathscr{A})$), i.e. Van Damme's result is true for the larger class of 2-person games with polytopal strategy spaces and bilinear payoff functions.

Proof of Proposition 2. Having these results, the proof of Proposition 2 boils down to showing two facts:
(i) (u, v) is an undominated equilibrium of $G(\mathscr{A})$ if and only if $(\mu(u), \nu(v))$ is an undominated equilibrium in $\Gamma(\mathscr{A})$.
(ii) (x, y) is a perfect equilibrium in $\mathscr{A}$ (according to our definition) if and only if (x, y) is perfect in $\Gamma(\mathscr{A})$.

Fact (i) is immediately clear. If $\bar{u}$ dominates u in $G(\mathscr{A})$, $\mu(\bar{u})$ dominates $\mu(u)$ in $\Gamma(\mathscr{A})$ and conversely if $\bar{x}$ dominates x in $\Gamma(\mathscr{A})$, then $\overline{\mu}(\bar{x})$ dominates $\overline{\mu}(x)$ in $G(\mathscr{A})$.

If (x, y) is a perfect equilibrium in $\mathscr{A}$ (according to our definition), V is an open neighborhood of (x, y) and δ is a number greater than zero, there is a perturbation $\mathscr{A}(\bar{x}_0, \bar{y}_0, \eta)$ of $\mathscr{A}$, with $\eta < \delta$ and $E(\mathscr{A}(\bar{x}_0, \bar{y}_0, \eta)) \cap V \neq \emptyset$. It is obvious that $\mathscr{A}(\bar{x}_0, \bar{y}_0, \eta)$ can be understood as an amalgation of games [4] and that $\Gamma(\mathscr{A}(\bar{x}_0, \bar{y}_0, \eta)) = \Gamma(\mathscr{A})(\bar{x}_0, \bar{y}_0, \eta)$. Notice that in $\Gamma(\mathscr{A})$ the points $\bar{x}_0$ and $\bar{y}_0$ are to be considered as one point in the relative interior of $\prod_{i \in M} \Delta(X^i)$ and $\prod_{j \in N} \Delta(Y^j)$, respectively. As $E(\mathscr{A}) = E(\Gamma(\mathscr{A}))$ for all amalgation of games (Proposition 1) we have $E(\Gamma(\mathscr{A})(\bar{x}_0, y_0, \eta)) \cap V \neq \emptyset$. The proof in the opposite direction uses the same arguments. □

In the following example we show that PF($\mathscr{A}$) can be a proper subset of UN($\mathscr{A}$).

Example. Let the player set be $M = \{a, b\}$ and $N = \{c\}$. The private conflicts of a and b against c are given by the matrix games

	L	M	R
U	2	2	1
D	1	6	1

and

	L	M	R
u	2	2	1
d	6	1	1

respectively. Then the matrix game $G(\mathscr{A})$ has the payoff matrix

	L	M	R
Uu	4	4	2
Ud	8	3	2
Du	3	8	2
Dd	7	7	2

In $G(\mathscr{A})$ the strategy Dd dominates Uu, but in $\mathscr{A}$ neither player (a) nor player (b) has a dominated strategy. The equilibrium set of $G(\mathscr{A})$ is $\Delta^4 \times \{R\}$ and the set of perfect (= undominated) equilibria consists of the two line segments [Ud, Dd] $\times$ {R} and [Du, Dd] $\times$ {R}. Hence, with proposition 2, the set of perfect equilibria of $\mathscr{A}$ consists of $\Delta^2 \times \{d\} \times \{R\}$ and $\{D\} \times \Delta^2 \times \{R\}$ and this is a proper subset of UN($\mathscr{A}$) = $E(\mathscr{A}) = \Delta^2 \times \Delta^2 \times \{R\}$ (by Proposition 1).

4. The structure of the set of (perfect) equilibria of amalgations of games

In a finite n-person strategic game Γ a *Nash component* is defined as a maximal convex subset of $E(\Gamma)$ and a *Selten component* as a maximal convex subset of PF(Γ). For n-matrix games with $n \geq 3$ these

[4] The strategy spaces are simplices and the payoff functions are multilinear. Each players' payoff is only dependent on his own strategies and the strategies of his opponents.

subsets are, in general, not very characteristic for the structure of the set of (perfect) equilibria but for *bimatrix games* they are. In fact, the following properties of Nash components are proved in Jansen (1981) (cf. also Heuer and Millham, 1976):

(i) *If N is a Nash component of a bimatrix game Γ, then $N = \mathrm{proj}_1(N) \times \mathrm{proj}_2(N)$ wherein proj_i is the projection of $\Delta_1 \times \Delta_2$ to Δ_i.*
(ii) *The Nash components of Γ are (products of)* polytopes.
(iii) *There are finitely many Nash components and the equilibrium set $E(\Gamma)$ is the irredundant union of these finitely many Nash components.*

For Selten components analogous properties hold:

(i)′ *If T is a Selten component of Γ, then $T = \mathrm{proj}_1(T) \times \mathrm{proj}_2(T)$.*

Proof. As $T \subset E(\Gamma)$ and T is convex, there is a Nash component $N = N_1 \times N_2$ with $T \subset N$. Then $\mathrm{proj}_i(T) \subset N_i$, $(i = 1, 2)$ and therefore, $\mathrm{proj}_1(T) \times \mathrm{proj}_2(T) \subset E(\Gamma)$ and $\mathrm{proj}_i(T)$, $i = 1, 2$ consist of undominated equilibria only. Then $\mathrm{proj}_1(T) \times \mathrm{proj}_2(T)$ consists of perfect equilibria,is convex and contains T. The maximality of T gives the result. □

(ii)′ *The Selten components of Γ are products of polytopes.*
(iii)′ *There are finitely many Selten components and the set of perfect equilibria of Γ is an irredundant union of these finitely many Selten components.*

The proof of (ii)′ and (iii)′ can be found in Borm et al. (1993). We proved property (i)′ because in Borm et al. (1993) Selten components are defined as subsets of $\mathrm{PF}(\Gamma)$ which are maximal among the subsets which are convex and *exchangeable*, i.e., have a product structure.

Using the results of Sections 2 and 3 we can prove the same results for amalgation of games.

Proposition 3. *An amalgation of games $\mathscr{A}$ has finitely many Nash components and Selten components. These components are polytopes and $E(\mathscr{A})$ (respectively $\mathrm{PF}(\mathscr{A})$) is an irredundant union of the Nash (respectively Selten) components. The Nash components (Selten components) are products of their projections to $\Delta_{[M]}$ and to $\Delta_{[N]}$.*

Proof. We give a proof for Nash components only. For Selten components the proof is obtained by using Proposition 2 at places where we now use Proposition 1.

As $G(\mathscr{A})$ is a bimatrix game, $E(G(\mathscr{A}))$ is the irredundant union of finitely many Nash components: $\bigcup_\alpha N_\alpha = \bigcup_\alpha P_\alpha \times Q_\alpha$. From Proposition 1 we infer that $E(\Gamma(\mathscr{A})) = \bigcup_\alpha(\mu(P_\alpha) \times \nu(Q_\alpha))$. As μ and ν are linear, $E(\Gamma(\mathscr{A}))$ is a finite union of products of polytopes. If there is a convex set $\tilde{N}$ with $\mu(P_\alpha) \times \nu(Q_\alpha) \subset \tilde{N} \subset E(\Gamma(\mathscr{A}))$ for some index α, the set $\{(u, v) | (\mu(u), \nu(v)) \in \tilde{N}\}$ is convex, contains $P_\alpha \times Q_\alpha$ and is a subset of $E(G(\mathscr{A}))$ by Proposition 1. It cannot be strictly larger by the maximality of $P_\alpha \times Q_\alpha$ and therefore, $\tilde{N} = \mu(P_\alpha) \times \nu(Q_\alpha)$.

The irredundance of $\{\mu(P_\alpha) \times \nu(Q_\alpha)\}$ (no component can be disposed of in covering $E(\Gamma(\mathscr{A}))$) follows from the irredundance of $\{P_\alpha \times Q_\alpha\}$ in covering $E(G(\mathscr{A}))$. □

In general it is not true that a Nash (or Selten) component of $\mathscr{A}$ is the product of the projections to the strategy sets of the players (see the next section for an example).

5. Elaboration of some examples

In this section we take a closer look to the examples in the introduction.

The tax inspector

We use the notation of the introduction. Let P be a population of tax payers and let $\{P^i \mid 1 \leq i \leq t\}$ be a partition of P into tax payers with the same amount c_i of tax to be evaded, $1 \leq i \leq t$. We assume $c_1 < c_2 < \cdots < c_t$. The number of tax payers of type i is p_i, $(p := \sum_{i=1}^{t} p_i)$. The fine to be paid by a tax payer when he is detected, is a function of c: $f = f(c)$. We assume that $\lambda(c) := f(c)/c$ is a strictly increasing function of c. The private conflicts have the following payoff matrices:

$$\begin{array}{ccc} & \text{C} & \text{N} \\ \text{D} & -f(c_i),\ f(c_i) - \gamma & c_i,\ -c_i \\ \text{H} & 0,\ -\gamma & 0,\ 0 \end{array}$$

Recall that γ are the checking cost.

In the bimatrix game $G(\mathscr{A})$ player P's (the population of tax payers) pure strategies can be identified with $(R_1, R_2, \ldots, R_t)$ wherein $R_i \subset P^i$ is the set of players of type i who are Dishonest. For player P (the tax payers as a group) and the tax inspector T only the numbers $r_i := |R_i|$ are relevant.

We follow the combinatorial–geometric method of Borm, Gijsberts and Tijs (1989) (see also Borm, 1992) to compute the Nash equilibria of the bimatrix game $G(\mathscr{A})$.

(A) *Computation of the best responses to a strategy q_c of the tax inspector.* If $q = q_c$ is the inspection rate of the tax inspector, the pure strategy $(r_1, \ldots, r_t)$ of player P gives a payoff $\sum_{i=1}^{t} r_i(c_i - q(f(c_i) + c_i))$ to P. It is a best response if

$$r_i = \begin{cases} p_i & \text{if } c_i/(f(c_i) + c_i) > q, \\ 0 & \text{if } c_i/(f(c_i) + c_i) < q, \end{cases} \qquad \text{and} \quad 0 \leq r_i \leq p_i \quad \text{if } c_i/(f(c_i) + c_i) = q.$$

So, the q-interval [0, 1] is divided into $t + 1$ segments

$$0 < 1/(1 + \lambda_t) < 1/(1 + \lambda_{t-1}) < \cdots < 1/(1 + \lambda_1) < 1$$

and we have: If $1/(1 + \lambda(c_s)) < q < 1/(1 + \lambda(c_{s-1}))$ for some s, there is a unique best reply of player P to q, namely $(p_1, \ldots, p_{s-1}, 0, \ldots 0)$. If $q = 1/(1 + \lambda(c_s))$ for some s, the tax payers of type s are indifferent between being Honest or Dishonest. The pure best responses are

$$(p_1, \ldots, p_{s-1}, r, 0, \ldots 0),\ 0 \leq r \leq p_s.$$

(B) *Labeling of the pure strategies of the tax payers.* The label of a strategy $(r_1, \ldots, r_t)$ gives the pure best responses of T to the strategy $(r_1, \ldots, r_t)$:

$$[\text{C}] \quad \text{if } \sum_{i=1}^{t} r_i(f(c_i) + c_i) > p\gamma,$$

$$[\text{CN}] \quad \text{if } \sum_{i=1}^{t} r_i(f(c_i) + c_i) = p\gamma,$$

$$[\text{N}] \quad \text{if } \sum_{i=1}^{t} r_i(f(c_i) + c_i) < p\gamma.$$

This means that the strategy $r_i = 0$ for $i = 1, \ldots, t$ has label [N] (if everybody is honest it makes no sense to make expenses for checking). If also the strategy $(p_1, \ldots, p_t)$ has label [N], all pure strategies have label [N] and $((p_1, \ldots, p_t), (0, 1))$ is the unique equilibrium (everybody is Dishonest but the checking cost are higher than the money that can be collected by the tax inspector). Therefore, we may assume that the pure strategy $(p_1, \dots, p_t)$ has label [C] or [CN]. Let s be the smallest index such that $\Sigma_{i=1}^{s} p_i(f(c_i) + c_i) \geq p\gamma$. First we consider the generic case with $\Sigma_{i=1}^{s} p_i(f(c_i) + c_i) > p\gamma$. Then only the strategy $q = 1/(1 + \lambda(c_s))$ has as well best responses with label [C] as best responses with label [N]: $(p_1, p_2, \dots, p_s, 0, \dots, 0)$ has label [C] and the strategy $(p_1, \dots, p_{s-1}, 0, \dots, 0)$ has label [N]. For $q = 1/(1 + \lambda(c_s))$ the strategies $(p_1, \dots, p_{s-1}, r, \dots, 0)$ with $0 \leq r \leq p_s$ are the best responses. In this case the equilibrium set consists of those mixtures

$$a = \sum_{r=0}^{p_s} \alpha_r (p_1, \dots, p_{s-1}, r, \dots, 0), \quad \alpha_r \geq 0, \quad \sum_{r=0}^{p_s} \alpha_r = 1,$$

that have both pure strategies C and N as best response, combined with $q = 1/(1 + \lambda(c_s))$ for T. Notice that the average number of tax payers of type s playing strategy Dishonest (i.e. $\Sigma_{r=0}^{p_s} \alpha_r r = \bar{r}_s$) is the same for every equilibrium strategy of P. So, we obtain a unique equilibrium strategy for the tax inspector and a lot of associated equilibrium strategies for the tax payers.

In the non-generic case (i.e. if $\Sigma_{i=1}^{s} p_i(f(c_i) + c_i) = p\gamma$), only the strategy $(p_1, \dots, p_s, \dots, 0)$ has label [NC] and it is the unique best response to any q between $1/(1 + \lambda_{s+1})$ and $1/(1 + \lambda_s)$. Then $a = (p_1, \dots, p_s, 0, \dots, 0)$ is the unique equilibrium strategy for the tax payers and $q \in [1/(1 + \lambda_{s+1}), 1/(1 + \lambda_s)]$ are the equilibrium strategies of the tax inspector.

Let $\bar{r}_s$ be the average number of tax payers of type s that plays, in equilibrium, the strategy D and let q be an equilibrium strategy of T. The payoff to T is the sum of $-q(\Sigma_{i=1}^{t} p_i)\gamma$ (the checking cost) and $\Sigma_{i<s} p_i(qf(c_i) - (1-q)c_i)) + \bar{r}_s(qf(c_s) - (1-q)c_s$ (the collected money). As

$$\left(\sum_{i=1}^{t} p_i \right) \gamma = \sum_{i<s} p_i(f(c_i) + c_i) + \bar{r}_s(f(c_s) + c_s),$$

we find for the equilibrium payoff

$$-\sum_{i<s} p_i c_i - \bar{r}_s c_s$$

(independent of q). A similar calculation gives the equilibrium payoff for P:

$$\sum_{i<s} p_i c_i + \bar{r}_s c_s - q \left(\sum_{i=1}^{t} p_i \right) \gamma.$$

(The last term is the checking cost).

As expected, the payoff for the tax payers has increased compared with the sum of the payoffs in the private conflicts but also – what is more characteristic – their strategies are 'de-randomized'; only players in the 'critical class' P^s (in fact only one player of this class) have to mix their (his) strategies. Notice that, in the generic case, the tax inspector focusses his attention to the tax payers in the 'critical' class P_s.

Prey contra predator

Let us consider an example of the prey contra predator model. Each species has two pure strategies E and L. The strategy E (or L) means that the animal is active, hunting or foraging, Early (or Late) in the day. Let the payoffs be:

		Wolves		Eagles	
		E	L	e	l
Rabbits	E	−1, 3	1, 0	−1, 2	1, 0
	L	1, 0	−1, 3	1, 0	−1, 2
Mice	e	−1, 1	1, 0	−1, 2	1, 0
	l	1, 0	−1, 1	1, 0	−1, 2

The game $G(\mathscr{A})$ has four pure strategies for both players: Ee, El, Le and Ll and the payoff matrices are:

	Ee	El	Le	Ll
Ee	−4, 8	0, 4	0, 4	4, 0
El	0, 5	0, 5	0, 3	0, 3
Le	0, 3	0, 3	0, 5	0, 5
Ll	4, 0	0, 4	0, 4	−4, 8

The game $G(\mathscr{A})$ has three Nash components: $\{p \mid p_2 = p_3\} \times \{q \mid q_1 = q_4\}$ and the isolated pure equilibria El × El and Le × Le. Notice that the first Nash component is not equal to the product of its projections to the strategy sets of the players of $\mathscr{A}$.

In the private conflicts all animals have to play the mixed strategy $(\frac{1}{2}, \frac{1}{2})$ and the total payoff is (0, 4). This is also the payoff in the first Nash component of $G(\mathscr{A})$. There are however two other interesting equilibria in *pure strategies* with payoff (0, 5), higher for the predators and equal for the preys. Moreover, in these equilibria the rabbits are 'matched' with the wolves and the mice with the eagles.

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